

Matrices over residuated lattices

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Residuated lattices simultaneously generalise lattice-ordered groups and the structures used to model various many-valued logics (e.g., Boolean algebras, Heyting algebras and MV-algebras). Matrices over residuated lattices are of interest in data analysis (specifically formal concept analysis) and tropical mathematics, and the linear algebra of such matrices can be surprisingly similar to that of matrices over a field. I will describe some of the ways in which linear algebra over residuated lattices is like—and is unlike—linear algebra over fields.

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1 Residuated lattices and matrices

Definition 1 A *residuated lattice* is a lattice-ordered monoid $(S, \cdot, 1)$ with the property that for each $x \in S$ the multiplication functions $x-: S \rightarrow S$ and $-x: S \rightarrow S$ have upper adjoints $d_R(x, -): S \rightarrow S$ and $d_L(-, x): S \rightarrow S$ respectively.

In other words, a lattice-ordered monoid $(S, \cdot, 1)$ is a residuated lattice if and only if there are two functions $d_R, d_L: S \times S \rightarrow S$ satisfying

$$xa \leq y \quad \Leftrightarrow \quad a \leq d_R(x, y) \tag{1}$$

and

$$ax \leq y \Leftrightarrow a \leq d_L(y, x) \quad (2)$$

for all $a, x, y \in S$. These conditions immediately tell us that $xd_R(x, y) \leq y$ and $d_L(y, x)x \leq y$ for all $x, y \in S$, and so in some sense the functions d_R and d_L approximate division in S . For this reason it is fairly standard to write $x \setminus y$ instead of $d_R(x, y)$, and y / x instead of $d_L(y, x)$.

Example 2 Let $(S, \cdot, 1)$ be a lattice-ordered group (for instance $(\mathbf{Z}, +, 0)$ with the usual ordering of integers). Then

$$xa \leq y \Leftrightarrow a \leq x^{-1}y \quad (3)$$

and

$$ax \leq y \Leftrightarrow a \leq yx^{-1} \quad (4)$$

for all $a, x, y \in S$, and as such S is a residuated lattice with $d_R(x, y) = x^{-1}y$ and $d_L(y, x) = yx^{-1}$ for all $x, y \in S$.

Example 3 Let X be a set and take $S = (\text{Pow}(X), \cap, X)$ with the usual ordering of subsets. Then

$$V \cap U \subseteq W \Leftrightarrow U \subseteq (X \setminus V) \cup W \quad (5)$$

for all $U, V, W \subseteq X$, and as such S is a residuated lattice with $d_R(V, W) = (X \setminus V) \cup W$ for all $V, W \in S$. Notice that d_R and d_L are identical in this example because S is a commutative monoid.

Example 4 Let $(M, \cdot, 1)$ be a monoid and take $S = (\text{Pow}(M), \cdot, \{1\})$ with the usual ordering of subsets. Then

$$VU = \{ba : b \in V, a \in U\} \subseteq W \Leftrightarrow U \subseteq \{a \in M : Va \subseteq W\} \quad (6)$$

and

$$UV \subseteq W \Leftrightarrow U \subseteq \{a \in M : aV \subseteq W\} \quad (7)$$

for all $U, V, W \subseteq M$, and as such S is a residuated lattice with $d_R(V, W) = \{a \in M : Va \subseteq W\}$ and $d_L(W, V) = \{a \in M : aV \subseteq W\}$ for all $V, W \in S$.

Any residuated lattice S can be viewed as a semiring¹ by taking “addition” and “multiplication” to be the join (\vee) and monoid (\cdot) operations on S respectively,² and, in short, this means that we have a way to add and multiply matrices with entries in S . Moreover, the fact that S also has a meet (\wedge) operation allows us to consider d_R and d_L as functions of matrices. That is, (1) and (2) still hold if ‘ a ’, ‘ x ’ and ‘ y ’ are taken to be matrices of appropriate sizes instead of scalars.

Problem 5 Let S be a semiring and let $A \in S^{m \times n}$ be an m -by- n matrix over S .

- (i) When do two column vectors $u, u' \in S^{n \times 1}$ give rise to the same element $Au = Au'$ of the column space of A ? In other words, describe the (set-theoretic) kernel of the surjective function $S^{n \times 1} \rightarrow \text{Col}(A)$ given by $u \mapsto Au$. Similarly for the row space of A .
- (ii) Does each left S -linear function $\text{Row}(A) \rightarrow S$ extend to a left S -linear function $S^{1 \times n} \rightarrow S$? Or, equivalently, to what extent is the injective function $\text{Col}(A) \rightarrow \text{Row}(A)^*$ given by $Au \mapsto (vA \mapsto vAu)$ surjective? Similarly for right S -linear functions on the column space of A .
- (iii) What, if any, is the relationship between the column space and row space of A ?

The diagram

$$\begin{array}{ccccc}
 S^{n \times 1} & \xrightarrow{u \mapsto Au} & \text{Col}(A) & \xrightarrow{Au \mapsto (vA \mapsto vAu)} & \text{Row}(A)^* \\
 & & \uparrow \text{---} \downarrow & & \\
 S^{1 \times m} & \xrightarrow{v \mapsto vA} & \text{Row}(A) & \xrightarrow{vA \mapsto (Au \mapsto vAu)} & \text{Col}(A)^*
 \end{array} \tag{8}$$

summarises the functions (and potential functions) of interest.

¹A ‘semiring’ is often required to have an additive identity element, but a residuated lattice need not have one (the lattice might not have a bottom element). However, residuated lattices do have “small enough” elements that—for finite sets, at least—behave exactly like an additive identity element. See Wilding et al. [6] or Hollings and Kambites [3].

²Definition 1 does not explicitly require the monoid operation to distribute over binary joins; this follows from the assumption that for each $x \in S$ the multiplication functions $x-$ and $-x$ have upper adjoints.

The questions posed in Problem 5 are easily answered in the case S is a field and $A \in S^{m \times n}$ is a matrix over S :

- (i) the kernel of the function $u \mapsto Au$ is determined by the class $\{u \in S^{n \times 1} : Au = 0\}$, which is a vector space isomorphic to $S^{1 \times n} / \text{Row}(A)$;
- (ii) each S -linear function $\text{Row}(A) \rightarrow S$ extends to an S -linear function $S^{1 \times n} \rightarrow S$;
- (iii) $\text{Col}(A) \cong \text{Row}(A)$ vector spaces.

2 Kernel classes

Now let S be a residuated lattice and let $A \in S^{m \times n}$. In contrast to the case of field, the kernel of the function $S^{n \times 1} \rightarrow \text{Col}(A)$ given by $u \mapsto Au$ will not be determined by a single class (because S does not have additive inverses). Instead, the structure of the class $\{u \in S^{n \times 1} : Au = x\}$ will depend upon which $x \in \text{Col}(A)$ is chosen. The result that describes the precise structure of these classes is rather technical, but it can be intuitively summarised as follows.

For $x \in \text{Col}(A)$, the class $\{u \in S^{n \times 1} : Au = x\}$ comprises all the vectors beneath $d_R(A, x)$ that do not belong to a different class, i.e., that are not also beneath a smaller $d_R(A, x')$.

In particular, the class corresponding to $x \in \text{Col}(A)$ has top element $d_R(A, x)$.

Example 6 Let $G = \{1, g, g^2\}$ denote the group of order 3 and consider $\{1, g\} \in S$, where $S = (\text{Pow}(G), \cdot, \{1\})$ is the residuated lattice defined in Example 4. By computing the function $S \rightarrow S$ given by $U \mapsto \{1, g\}U$, we deduce (see Figure 1) that

$$\text{Col}(\{1, g\}) = \{\emptyset, \{1, g\}, \{1, g^2\}, \{g, g^2\}, G\}. \quad (9)$$

We also observe that the top elements of the classes of the kernel of this function are \emptyset , $\{1\}$, $\{g^2\}$, $\{g\}$ and G .

Since we chose join to play the role of “addition” in S , the column space of A is automatically closed under taking joins in $S^{m \times 1}$. In general it is not closed under taking meets in $S^{m \times 1}$, however. Yet it turns out that $\text{Col}(A)$ is still a lattice; meets in $\text{Col}(A)$ just need not necessarily coincide with meets in $S^{m \times 1}$. The trick is to take meets in $S^{n \times 1}$ by first applying the function $d_R(A, -)$, and then multiply by A to get back into $\text{Col}(A)$. That is, the meet of x_1, x_2 in $\text{Col}(A)$ is $A(d_R(A, x_1) \wedge d_R(A, x_2))$.

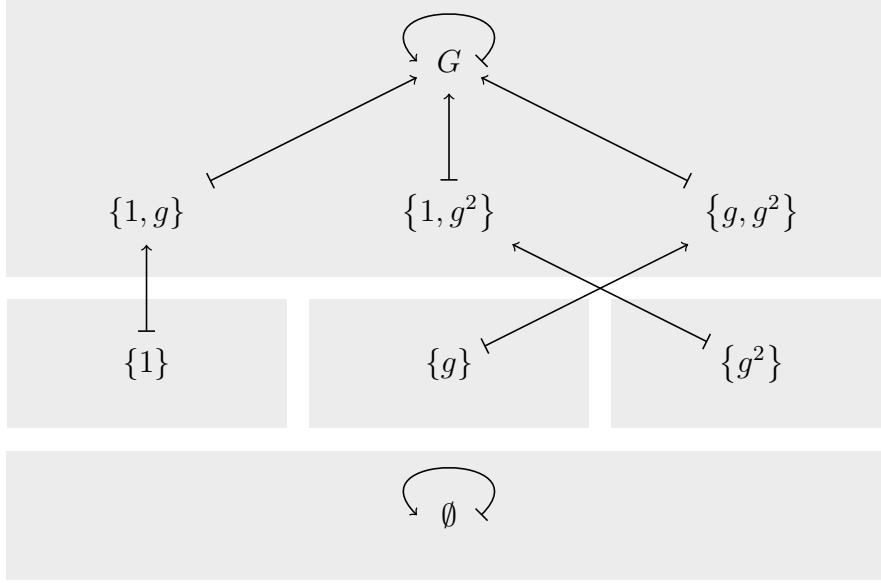


Figure 1: The function $S \rightarrow S$ given by $U \mapsto \{1, g\}U$ in Example 6, with shaded kernel classes.

3 Extension of linear functions

Theorem 7 *Let M be a finite monoid and take $S = (\text{Pow}(M), \cdot, \{1\})$. For each $a \in M$ there is some $A \in S^{2 \times 2}$ and some left S -linear function $f: \text{Row}(A) \rightarrow S$ with the property that f extends to a left S -linear function $S^{1 \times 2} \rightarrow S$ if and only if a has a right inverse in M .*

Proof (sketch) Take

$$A = \begin{bmatrix} \emptyset & \{a^k\} \\ a & \{1\} \end{bmatrix}, \quad (10)$$

where $k \in \mathbf{N}$ is chosen such that a^k is idempotent (this is possible because M is finite; see Howie [4, Proposition 1.2.3]), and define $f: \text{Row}(A) \rightarrow S$ by

$$f\left(\begin{bmatrix} U & V \end{bmatrix} A\right) = VM \quad (11)$$

for all $U, V \in S$. This function extends if and only if

$$\begin{bmatrix} \emptyset \\ M \end{bmatrix} = A \begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} a^k X \\ aW \cup X \end{bmatrix} \quad (12)$$

for some $W, X \in S$, which happens if and only if $M = aW$ for some $W \in S$. Hence f extends if and only if there is some $b \in M$ satisfying $1 = ab$. \square

Theorem 7 tells us that not every residuated lattice has the field-like property that each left S -linear function from the row space of a matrix to S extends, but it also leaves open the possibility that some special residuated lattices (e.g., $(\text{Pow}(G), \cdot, \{1\})$ for a group G) might actually have this property.

Definition 8 A residuated lattice $(S, \cdot, 1)$ is *involutive* (see Wille [7]) if there is an involution $\bar{} : S \rightarrow S$ satisfying $\bar{a} = d_R(a, \bar{1})$ and $\bar{a} = d_L(\bar{1}, a)$ for all $a \in S$.

Example 9 (cf. Example 2) Let $(S, \cdot, 1)$ be a lattice-ordered group. Then S is an involutive residuated lattice with $\bar{a} = a^{-1}$ for all $a \in S$.

Example 10 (cf. Example 3) Let X be a set and take $S = (\text{Pow}(X), \cap, X)$. Then S is an involutive residuated lattice with $\bar{U} = X \setminus U$ for all $U \in S$.

Example 11 (cf. Example 4) Let $(G, \cdot, 1)$ be a group and take $S = (\text{Pow}(G), \cdot, \{1\})$. Then S is an involutive residuated lattice with $\bar{U} = G \setminus U^{-1}$ for all $U \in S$, where $U^{-1} = \{a^{-1} : a \in U\}$.

If S is an involutive residuated lattice then we can define an involution on the set of matrices over S by setting $\overline{A_{ij}} = \overline{A_{ji}}$ for each $A \in S^{m \times n}$. This allows us to prove the following result.

Theorem 12 Let S be an involutive residuated lattice and let $A \in S^{m \times n}$. Then each left S -linear function $\text{Row}(A) \rightarrow S$ extends to a left S -linear function $S^{1 \times n} \rightarrow S$.

Proof (sketch) Let $f : \text{Row}(A) \rightarrow S$ be left S -linear. Then we have $f(vA) = vA\overline{fAA}$ for all $v \in S^{1 \times m}$, where $fA \in S^{m \times 1}$ denotes the column vector obtained by applying f to the m rows of A . \square

4 Column spaces vs. row spaces

The involution just defined for matrices over a residuated lattice S also gives us a way to pass between the column space and row space of a matrix $A \in S^{m \times n}$. Specifically, the functions $\text{Col}(A) \rightarrow \text{Row}(A)$ and $\text{Row}(A) \rightarrow \text{Col}(A)$ given by $x \mapsto \bar{x}A$ and $y \mapsto A\bar{y}$ respectively are inverses of each other, so constitute a bijection between $\text{Col}(A)$ and $\text{Row}(A)$. Moreover, each of these functions is order-reversing, and thus we conclude that $\text{Col}(A)$ and $\text{Row}(A)$ are anti-isomorphic lattices.¹

¹In fact, $\text{Col}(A)$ and $\text{Row}(A)$ are anti-isomorphic in an even stronger sense. See Wilding et al. [6]

References

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