The illustrated zoo of order-preserving functions

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`http://dpw.me/mathematics/`

Posets (partially ordered sets) underlie much of mathematics, but we often don't give them a second thought. I will describe some of the interesting posets and, more importantly, order-preserving functions between posets that turn up in various branches of mathematics. The order-preserving functions on show will range all the way from the very basic (monotone functions) to the most powerful (residuated functions and order isomorphisms).

Contents

1 Introduction

The notion of a partial order (a reflexive, anti-symmetric and transitive binary relation) goes back at least as far as Leibniz in or around 1690. Of course, Leibniz did not give the modern formulation, but a careful reading of his work reveals the key ideas, such as transitivity.

> In the same way, all men are contained in all animals, and all animals in all corporeal substances; therefore all men are contained in corporeal substances.
>
> (Leibniz [2])

In fact, Leibniz essentially defined what is now called a join-semilattice.

Definition 1 A poset X is a *join-semilattice* if every pair of elements has a least upper bound in X. That is, for all $x_1, x_2 \in X$ there should be some element $x_1 \vee x_2 \in X$ (called the *join* of x_1 and x_2) with $x_1, x_2 \leq x_1 \vee x_2$ and $x_1 \vee x_2 \leq x$ for all $x \in X$ satisfying $x_1, x_2 \leq x$.

If every pair of elements in a poset X has a greatest lower bound (denoted $x_1 \wedge x_2$ for $x_1, x_2 \in X$ and defined dually to least upper bound) then X is called a *meet-semilattice*, and if X is both a join- and a meet-semilattice then it is simply called a *lattice*. The most familiar lattice is probably the powerset of a set. In this lattice the partial order is subset inclusion, the join of a pair of subsets is their union and the meet of a pair of subsets is their intersection.

It is common to draw posets as elements connected by lines, with the convention that an element x_2 is (strictly) greater than an element x_1 in the partial order if and only there is an upwards path from x_1 to x_2. For example, the powerset of $\{0, 1, 2\}$ is shown in this way in Figure 1. Such a representation of a poset is called a *Hasse diagram*.

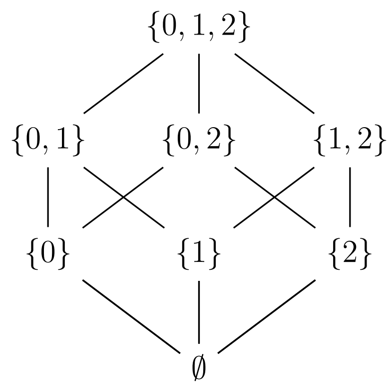

Figure 1: The powerset of $\{0, 1, 2\}$.

2 Monotone functions and order isomorphisms

Posets are sets with a single binary relation (the order), so it makes sense to ask that functions between posets respect this relation. In other words, we should like to work in the category **Pos** whose objects are posets and whose morphisms are monotone functions between posets.

Definition 2 Let X and Y be posets. A function $f\colon X \to Y$ is *monotone* if

$$x_1 \leq x_2 \quad \Rightarrow \quad f(x_1) \leq f(x_2) \tag{1}$$

for all $x_1, x_2 \in X$.

A monotone function between posets need not respect any additional structure (such as joins and/or meets) that the posets may have. An order isomorphism, on the other hand, respects all structure; the existence of an order isomorphism between posets means that they are actually "the same" poset.

Definition 3 Let X and Y be posets. A function $f\colon X \to Y$ is an *order isomorphism* if f is a monotone bijection and f^{-1} is monotone.

The condition that f^{-1} be monotone in Definition 3 is required because the inverse of a monotone bijection is not automatically monotone. Indeed, Figure 2 shows a monotone bijection whose inverse is not monotone.

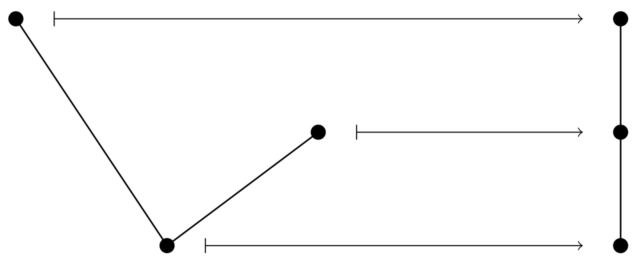

Figure 2: A monotone bijection between two 3-element posets.

3 Kernels and order embeddings

Any subset of a poset is again a poset, so in particular the image of a monotone function is a poset. The kernel of a monotone function $f\colon X \to Y$ is not a poset, however, but is instead an equivalence relation on X.

Definition 4 Let $f\colon X \to Y$ be monotone. The *kernel* of f is the equivalence relation $\ker(f) \subseteq X \times X$ defined by

$$(x_1, x_2) \in \ker(f) \quad \Leftrightarrow \quad f(x_1) = f(x_2) \tag{2}$$

for all $x_1, x_2 \in X$.

If $f\colon X \to Y$ is monotone then the set $X/\ker(f)$ of equivalence classes can be partially ordered by setting $[x_1] \leq [x_2]$ if and only if $f(x_1) \leq f(x_2)$. This is well-defined by (2). The injective function $g\colon X/\ker(f) \to Y$ given by $g\big([x]\big) = f(x)$ then satisfies

$$[x_1] \leq [x_2] \quad \Leftrightarrow \quad g\big([x_1]\big) \leq g\big([x_2]\big) \tag{3}$$

for all $[x_1], [x_2] \in X/\ker(f)$, so is (in particular) monotone. Moreover, the inverse of the bijection $g\colon X/\ker(f) \to \mathrm{im}(f)$ is also monotone, so $g\colon X/\ker(f) \to \mathrm{im}(f)$ is an order isomorphism. This proves the following "first isomorphism theorem" for posets.

Theorem 5 *If $f\colon X \to Y$ is monotone then $X/\ker(f)$ and $\mathrm{im}(f)$ are order isomorphic.*

Condition (3) motivates the definition of another type of order-preserving function.

Definition 6 Let X and Y be posets. A function $f\colon X \to Y$ is an *order embedding* if

$$x_1 \leq x_2 \quad \Leftrightarrow \quad f(x_1) \leq f(x_2) \tag{4}$$

for all $x_1, x_2 \in X$.

Order embeddings are, by definition, monotone. It is also clear that they are injective. However, order embeddings are more powerful than monotone injections because the domain and image of an order embedding are order isomorphic, whereas the domain and image of a monotone injection need not be.

4 Residuated functions

Residuated functions appear in many familiar and interesting settings because they respect much of a poset's structure, yet do not need to be order isomorphisms.

Definition 7 Let X and Y be posets. A function $f\colon X \to Y$ is *residuated* if there is a function $f_\wedge\colon Y \to X$ with

$$f(x) \leq y \quad \Leftrightarrow \quad x \leq f_\wedge(y) \tag{5}$$

for all $x \in X$ and all $y \in Y$.

If $f\colon X \to Y$ is residuated then the function $f_\wedge$ (called the *residual* of f) is unique and is given by

$$f_\wedge(y) = \max\{x \in X : f(x) \leq y\} \tag{6}$$

for all $y \in Y$. We use the notation '$f_\wedge$' because $f_\wedge$ sends meets to meets if X and Y are meet-semilattices. Similarly f sends joins to joins if X and Y are join-semilattices. It is an easy exercise to show that every residuated function is monotone, but the converse is not true because (as noted above) monotone functions need not respect joins.

Theorem 8 *If $f\colon X \to Y$ is a residuated injection then f is an order embedding.*

Proof Since f is monotone it suffices to show that

$$x_1 \leq x_2 \quad \Leftarrow \quad f(x_1) \leq f(x_2) \tag{7}$$

for all $x_1, x_2 \in X$. If $f(x_1) \leq f(x_2)$ then $x_1 \leq (f_\wedge \circ f)(x_2)$ by (5). A residuated function always satisfies $f \circ f_\wedge \circ f = f$, so since f is injective we have $f_\wedge \circ f = \mathrm{id}_X$. Hence $x_1 \leq (f_\wedge \circ f)(x_2) = x_2$ as required. □

The most obvious examples of residuated functions are order isomorphisms: if f is an order isomorphism then the residual of f is just f^{-1}. We now consider some less trivial examples.

Example 9 Let R be a commutative ring. A subgroup I of $(R, +, 0)$ is called an *ideal* of R if $Ib = \{ab : a \in I\} \subseteq I$ for all $b \in R$. Let X be the poset of ideals of R (ordered by subset inclusion) and let $I \in X$ be a fixed ideal of R. Define a function $f\colon X \to X$ by $f(J) = IJ$, where IJ denotes the ideal generated by the set $\{ab : a \in I, b \in J\}$. We then have

$$f(J) \subseteq K \quad \Leftrightarrow \quad J \subseteq \{b \in R : Ib \subseteq K\} \tag{8}$$

for all $J, K \in X$, and as such f is residuated with residual $f_\wedge\colon X \to X$ given by

$$f_\wedge(K) = \{b \in R : Ib \subseteq K\} \tag{9}$$

for all $K \in X$.

Example 10 Let H be a Hilbert space, let X be the poset of subspaces of H, let Y be the poset of closed subspaces of H (X and Y are both ordered by subset inclusion) and define $f\colon X \to Y$ by $f(M) = M^{\perp\perp}$. We then have

$$f(M) \subseteq N \quad \Leftrightarrow \quad M \subseteq N \tag{10}$$

for all $N \in Y$, and as such f is residuated with residual $f_\wedge\colon Y \to X$ given by $f_\wedge(N) = N$.

Example 11 A *Boolean algebra* is (in particular[1]) a lattice in which the join and meet operations distribute over one another and upon which a *complement* $\neg$ is defined, e.g., the powerset of a set. Let X be a Boolean algebra, fix $x \in X$ and define $f\colon X \to X$ by $f(y) = x \wedge y$. We then have

$$f(y) \leq x \quad \Leftrightarrow \quad y \leq \neg x \vee z \tag{11}$$

for all $y, z \in X$, and as such f is residuated with residual $f_\wedge\colon X \to X$ given by

$$f_\wedge(z) = \neg x \vee z = x \to z \tag{12}$$

for all $z \in X$.

5 Conclusion

Figure 3 summarises (in the style of a Hasse diagram) the relationships between the various types of order-preserving function that we have discussed above.

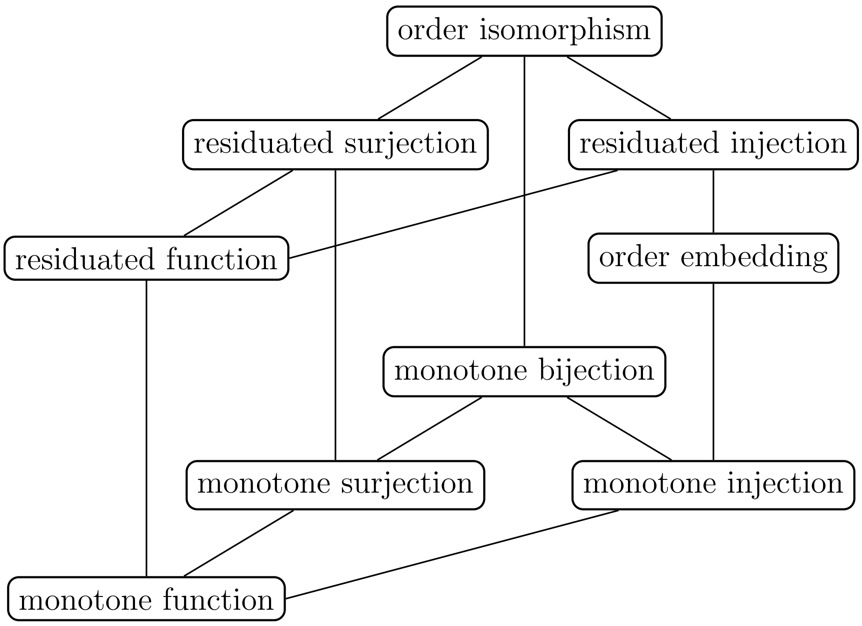

Figure 3: Order-preserving functions arranged by strength.

[1]A Boolean algebra must also have a greatest element and a least element.

References

[1] T. S. Blyth. *Lattices and Ordered Algebraic Structures.* Springer, London, 2005.

[2] G. W. Leibniz. A study in the calculus of real addition (after 1690). English translation in [3, pp. 131–144].

[3] G. H. R. Parkinson. *Leibniz: Logical Papers.* Clarendon Press, Oxford, 1966.